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Journal of Computational and Applied Mathematics 170 (2004) 197–216

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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The singular finite element method for some elliptic boundary value problem with interface☆

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Received 10 February 2003; received in revised form 13 January 2004

Abstract

A kind of numerical method is proposed for some elliptic boundary value problems with interface. It includes mainly two steps. At the first step, we obtain the approximation of the singularity near the singular points by solving a simple eigenvalue problem, which is one dimension less than the original problem. At the second step, we apply the approximation of the singularity together with the standard finite element basis functions to construct the singular finite element space and finally solve the original problem on a conventional mesh. Some numerical examples show the effectiveness of this method.

Keywords: Singularity; Finite element method; Interface

1. Introduction

Many physical phenomena can be modelled by partial differential equations with singularities. For example, the problem in a domain which is composed of several different materials usually corresponds to a partial differential equation with discontinuous coefficients. The solution of this interface problem may be singular [3,2,14,13,21]. The standard finite difference and finite element methods are difficult to give satisfactory numerical results for this kind of problems. Several improved methods have been proposed to increase the accuracy of the numerical solution. A natural method to treat the singularity is refining the mesh near the singularity [6,4,16,15,5], but it may be costly if the singularity is strong. As a different way to treat the singularity, in [10,11,9,12,19,18,20],

☆ The research is supported by RGC Grant of Hong Kong.

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0377-0427/$ - see front matter

doi:10.1016/j.cam.2004.01.002

the singular point is separated by using an artificial boundary, and a proper boundary condition is proposed on the artificial boundary. The original problem then can be reduced to a problem in a domain without singular points, and the normal finite element method can be applied. Givoli et al. [10,11,9,12,19,18] gave artificial boundary condition in analytical form. For the interface problems, the analytical form of the artificial boundary condition is difficult to obtain, Wu [20] proposed an artificial boundary condition in a discrete form. This method is effective but cannot obtain the numerical solution in the neighborhood of the singular point. Singular finite element method is another method to treat the singularity [1,8,17]. In this method, the singular basis functions are chosen to be the main terms in the singular expansion form of the exact solution. So that a high-order convergence can be reached, and more accurate numerical solutions can be expected with comparable computation work. In programming, the data construction and code, in comparison with the other two methods, are simpler because the mesh of the computational domain is of conventional type. The main disadvantage of this method is that an analytical singular expansion form of the exact solution must be known beforehand. Unfortunately, this is not the case for many problems, such as the interface problems.

In this paper, we follow the idea of singular finite element method, but eliminate the requirement for the analytical singular expansion form of the exact solution. We first obtain a discrete singular expansion near singular point by solving a simple eigenvalue problem, which is one dimension less than the original problem, and then use the approximation of the singularity together with the standard finite element basis functions to construct the singular finite element space. Finally, we find the singular finite element solution on a conventional mesh. Numerical examples show that this method is very effective.

In Section 2, we introduce the interface problem which is used to describe our method. Section 3 illustrates how to obtain the approximation of the singularity near the singular point. In Section 4, we construct the singular finite element space by using the approximation of the singularity and the standard finite element basis functions. In Section 5, two numerical examples are given to show the effectiveness of our method.

2. The interface problem

Let Ω be a bounded domain in R^2 and Γ be the boundary of Ω, as shown in Fig. 1(a). We assume that all the interfaces are straight lines and they meet at one point, which is assumed to be the origin. Under the polar coordinates, we assume that the interfaces have angles $\phi_1 = 0, \phi_2, \ldots, \phi_M$. The subdomains bounded by $\theta = \phi_k$ and $\theta = \phi_{k+1}, k = 1, 2, \ldots, M (\phi_{M+1} = \phi_1)$, are denoted by $\Omega_1, \Omega_2, \ldots, \Omega_M$. We consider the following interface problem

$$-\nabla(p\nabla u) = f \quad \text{in } \Omega, \tag{2.1}$$

$$u = g \quad \text{on } \Gamma, \tag{2.2}$$

$$u(r, \phi_k - 0) = u(r, \phi_k + 0), \quad 1 \leqslant k \leqslant M, \tag{2.3}$$

$$p_{k-1}\frac{\partial u}{\partial \mathbf{n}}(r, \phi_k - 0) = -p_k\frac{\partial u}{\partial \mathbf{n}}(r, \phi_k + 0), \quad 1 \leqslant k \leqslant M, \tag{2.4}$$

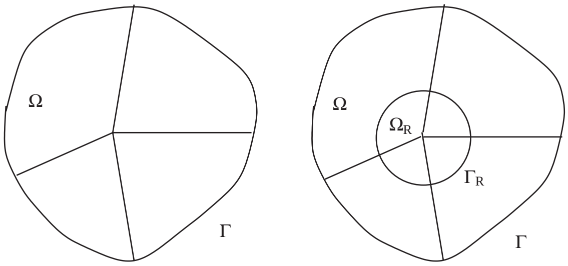

Fig. 1. (a) Domain Ω with interfaces; (b) Domain Ω_R in Ω.

where function p is a positive piecewise constant, namely, $p = p_k, x \in \Omega_k, k = 1, 2, \ldots, M,\ p_0 = p_M, f$ and g are two given functions on Ω and Γ, $\mathbf{n}$ is the outward unit normal to the interface.

Let $H^m(\Omega)$ and $H^\alpha(\Gamma)$ denote the usual Sobolev spaces on Ω and Γ, respectively. Introduce

$$H_g^1(\Omega) = \{v, v \in H^1(\Omega), v|_\Gamma = g\},$$

$$H_0^1(\Omega) = \{v, v \in H^1(\Omega), v|_\Gamma = 0\}.$$

Then the equivalent variational form of (2.1)–(2.4) is

Find $u \in H_g^1(\Omega)$ such that

$$a(u, v) = (f, v), \ \forall v \in H_0^1(\Omega), \tag{2.5}$$

where

$$a(u, v) = \int_\Omega p \nabla u \nabla v \, \mathrm{d}\mathbf{x}, \ (f, v) = \int_\Omega f v \, \mathrm{d}\mathbf{x}.$$

Clearly, the bilinear form $a(u, v)$ is bounded and coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$, i.e., there exists a constant $\alpha > 0$ such that

$$a(v, v) \geqslant \alpha \|v\|_{1,\Omega}^2, \ \forall v \in H_0^1(\Omega).$$

Thus we have the following theorem [20]:

Theorem 1. *For given $f \in H^{-1}(\Omega)$ and $g \in H^{-1/2}(\Gamma)$, variational problem* (2.5) *has a unique solution $u \in H_g^1(\Omega)$.*

Away from the cross point, the solution u is not in H^2, due to the jump in the normal derivative across the interface [13,14], but restricted to Ω_k, $u|_{\Omega_k},\ k = 1, 2, \ldots, M$, are in H^2. Generally, the exact solution of the interface problem can be expressed as

$$u(r, \theta) = c + a_1 r^{\lambda_1} f_1(\theta) + a_2 r^{\lambda_2} f_2(\theta) + \cdots \tag{2.6}$$

near the cross point, where c and $a_k, \lambda_k, k = 1, 2, \ldots,$ are constants, $f_j(\theta), j = 1, 2, \ldots,$ are piecewise smooth functions, $0 < \lambda_i \leqslant \lambda_j$, if $i < j$.

Naturally, if (2.6) can be approximated by

$$u(r, \theta) \approx \tilde{c} + \tilde{a}_1 r^{\tilde{\lambda}_1} \tilde{f}_1(\theta) + \tilde{a}_2 r^{\tilde{\lambda}_2} \tilde{f}_2(\theta) + \cdots \tag{2.7}$$

near the cross point, where $0 < \tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \ldots, \tilde{c}$ is a constant, coefficients $\tilde{a}_j, j = 1, 2, \ldots,$ may be unknown, then we can use $r^{\tilde{\lambda}_j} \tilde{f}_j(\theta),\ j = 1, 2, \ldots, m$, to construct a singular finite element space:

$$S_k^h = \mathrm{Span}\{\varphi_i(\mathbf{x}), \eta(r,\theta) r^{\tilde{\lambda}_j} \tilde{f}_j(\theta),\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\}, \tag{2.8}$$

where $\{\varphi_i(\mathbf{x})\}_1^n$ are the standard finite element basis functions which are the piecewise polynomials of degree k, $\eta(r,\theta)$ is a proper smooth cut-off function which equals one at the singular point and vanishes on the boundary Γ, m is a proper number matched up to $\{\varphi_i(\mathbf{x})\}_1^n$. For example, if $\varphi_i(\mathbf{x}), i = 1, 2, \ldots, n$, are the piecewise linear polynomials, according to the error estimation for the standard finite element method, we can take m such that

$$\tilde{\lambda}_{m+1} > 1, \text{ but } \tilde{\lambda}_m \leqslant 1. \tag{2.9}$$

It is plausible for us to expect that this kind of singular finite element approximation solution in S_k^h will be more accurate than the standard finite element approximation solution. In the next section we will describe the numerical method to obtain the approximation of the singularity.

3. The approximation of the singularity

Choose a small number R such that $\Gamma_R = \{\mathbf{x} : |\mathbf{x}| = R\} \subset \Omega$ and denote $\Omega_R = \{\mathbf{x} : |\mathbf{x}| < R\}$ (Fig. 1(b)). We consider the restriction of u, the solution of problem (2.1)–(2.4), on Ω_R. For simplicity, we assume that $f = 0$ in Ω_R. Then under the polar coordinates we have

$$\frac{\partial}{\partial r}\left(pr\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(p\frac{\partial u}{\partial \theta}\right) = 0 \quad \text{in } \Omega_R, \tag{3.1}$$

$$u|_{r=R} = u(R,\theta),\ u \text{ is bounded as } r \to 0, \tag{3.2}$$

$$u|_{\theta=\phi_k-0} = u|_{\theta=\phi_k+0}, \quad 1 \leqslant k \leqslant M, \tag{3.3}$$

$$p_{k-1}\left.\frac{\partial u}{\partial \theta}\right|_{\theta=\phi_k-0} = p_k\left.\frac{\partial u}{\partial \theta}\right|_{\theta=\phi_k+0}, \quad 1 \leqslant k \leqslant M, \tag{3.4}$$

where $u(R,\theta)$ is unknown, so problem (3.1)–(3.4) cannot be solved independently.

Let

$$V = \{v(\theta) : v(\theta) \in H^1(0, 2\pi), v(0) = v(2\pi)\},$$

$$U = \left\{u(r,\theta) : \text{ for fixed } r,\ 0 < r \leqslant R,\ u,\ \frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2} \in V\right\},$$

$$A_1(u,v) = \int_0^{2\pi} p(\theta) uv \,\mathrm{d}\theta, \quad A_2(u,v) = \int_0^{2\pi} p(\theta)\frac{\partial u}{\partial \theta}\frac{\mathrm{d}v}{\mathrm{d}\theta}\,\mathrm{d}\theta.$$

Then problem (3.1)–(3.4) is equivalent to the following differential-variational problem:

Find $u(r,\theta)\in U$ such that

$$A_1(r\partial_r(r\partial_r u), v) - A_2(u,v) = 0, \ \ \forall v \in V, \tag{3.5}$$

$$u|_{r=R} = u(R,\theta), \quad u \text{ is bounded as } r\to 0. \tag{3.6}$$

Now we choose $L+1$ nodes:

$$0=\theta_{l_1}<\theta_{l_2}<\cdots<\theta_{l_{L+1}}=2\pi,$$

such that $\{\phi_i\}_{i=1}^M \subset \{\theta_{l_j}\}_{j=1}^{L+1}$. Using these nodes, we partition interval $[0,2\pi]$ into L elements: $e_j=[\theta_{l_j},\theta_{l_{j+1}}]$, $j=1,2,\ldots,L$, and construct the finite element space:

$$V_k^{h_\theta} = \{v_{h_\theta}(\theta)\colon v_{h_\theta}(\theta)|_{e_j} \text{ is a polynomial of degree } k,\ j=1,2,\ldots,L\} \subset V,$$

where the Lagrange basis function $\psi_i(\theta)$ of $V_k^{h_\theta}$ satisfies

$$\psi_1(\theta_j)=\begin{cases}1 & j=1, j=N+1,\\ 0 & \text{otherwise,}\end{cases} \qquad \psi_i(\theta_j)=\begin{cases}1 & j=i,\\ 0 & \text{otherwise,}\end{cases} \qquad i=2,3,\ldots,N,$$

where $N=L\times k$, $\theta_j, j=1,2,\ldots,N+1$, are nodes satisfying

$$0=\theta_1<\theta_2<\cdots<\theta_{N+1}=2\pi,$$

and $h_\theta=\max_{1\leqslant j\leqslant N}|\theta_{j+1}-\theta_j|$. Let

$$U_k^{h_\theta}=\left\{u_{h_\theta}(r,\theta):\ \text{for fixed } r, 0<r\leqslant R, u_{h_\theta}, \frac{\partial u_{h_\theta}}{\partial r}, \frac{\partial^2 u_{h_\theta}}{\partial r^2}\in V_k^{h_\theta}\right\}\subset U.$$

Then the semi-discrete approximation of problem (3.5) and (3.6) is

Find $u_{h_\theta}(r,\theta)\in U_k^{h_\theta}$ such that

$$A_1(r\partial_r(r\partial_r u_{h_\theta}), v_{h_\theta}) - A_2(u_{h_\theta}, v_{h_\theta}) = 0, \quad \forall v_{h_\theta}\in V_k^{h_\theta}, \tag{3.7}$$

$$u_{h_\theta}|_{r=R}=u_{h_\theta}^0, \quad u_{h_\theta} \text{ is bounded as } r\to 0, \tag{3.8}$$

where $u_{h_\theta}^0=(\hat{\mathbf{u}}_{h_\theta}^0)^t\psi(\theta)$ with

$$\hat{\mathbf{u}}_{h_\theta}^0=[u(R,\theta_1),u(R,\theta_2),\ldots,u(R,\theta_N)]^t, \quad \psi(\theta)=[\psi_1(\theta),\psi_2(\theta),\ldots,\psi_N(\theta)]^t.$$

For any $u_{h_\theta}(r,\theta)\in U_k^{h_\theta}$, let

$$\hat{\mathbf{u}}_{h_\theta}(r)=[u_{h_\theta}(r,\theta_1),u_{h_\theta}(r,\theta_2),\ldots,u_{h_\theta}(r,\theta_N)]^t.$$

Then

$$u_{h_\theta}(r,\theta)=(\hat{\mathbf{u}}_{h_\theta}(r))^t\psi(\theta).$$

Thus problem (3.7) and (3.8) is equivalent to the following boundary value problem of a system of ordinary differential equations:

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}(B_1\hat{\mathbf{u}}_{h_\theta}(r))\right) - B_2\hat{\mathbf{u}}_{h_\theta}(r) = 0, \tag{3.9}$$

$$\hat{\mathbf{u}}_{h_\theta}(r)|_{r=R} = \hat{\mathbf{u}}^0_{h_\theta},\ \ \hat{\mathbf{u}}_{h_\theta}(r) \text{ is bounded as } r \to 0, \tag{3.10}$$

where B_1 and B_2 are two $N \times N$ matrices:

$$B_1 = \int_0^{2\pi} p(\theta)\psi(\theta)(\psi(\theta))^t\,\mathrm{d}\theta = \left(\int_0^{2\pi} p(\theta)\psi_i(\theta)\psi_j(\theta)\,\mathrm{d}\theta\right)_{N\times N},$$

$$B_2 = \int_0^{2\pi} p(\theta)\psi'(\theta)(\psi'(\theta))^t\,\mathrm{d}\theta = \left(\int_0^{2\pi} p(\theta)\psi_i'(\theta)\psi_j'(\theta)\,\mathrm{d}\theta\right)_{N\times N}.$$

Lemma 1. *Let γ_j and ξ_j, $j = 1,2,\ldots,N$, be the eigenvalues and the corresponding eigenvectors of the following eigenvalue problem*:

$$B_1^{-1}B_2\xi = \gamma\xi. \tag{3.11}$$

Then (3.9) *and* (3.10) *have the solution of the form*

$$\hat{\mathbf{u}}_{h_\theta}(r) = \sum_{j=1}^{N} \hat{a}_j r^{\hat{\lambda}_j}\xi_j, \tag{3.12}$$

where $\hat{\lambda}_j = \sqrt{\gamma_j}$, $j = 1,2,\ldots,N$, and $\hat{a}_j$, $j = 1,2,\ldots,N$, are constants satisfying

$$\sum_{j=1}^{N} \hat{a}_j R^{\hat{\lambda}_j}\xi_j = \hat{\mathbf{u}}^0_{h_\theta}. \tag{3.13}$$

Proof. Let $\hat{\mathbf{u}}_{h_\theta}(r) = r^{\hat{\lambda}}\xi$, where $\hat{\lambda} \in R$ and $\xi \in R^N$ are to be determined. Substituting $\hat{\mathbf{u}}_{h_\theta}(r)$ into system (3.9) and (3.10) we have

$$\hat{\lambda}^2 B_1\xi - B_2\xi = 0. \tag{3.14}$$

$\forall \mathbf{c} = [c_1, c_2, \ldots, c_N]^t \in R^N$, denote $v_{h_\theta}(\theta) = \mathbf{c}^t\psi(\theta) \in V_k^{h_\theta}$. For any $\mathbf{c} \neq 0$,

$$\mathbf{c}^t B_1 \mathbf{c} = \int_0^{2\pi} p(\theta)(v_{h_\theta}(\theta))^2\,\mathrm{d}\theta > 0, \quad \mathbf{c}^t B_2 \mathbf{c} = \int_0^{2\pi} p(\theta)(v'_{h_\theta}(\theta))^2\,\mathrm{d}\theta \geqslant 0,$$

so symmetric matrices B_1 and B_2 are positive definite and positive semi-definite, respectively, and there exists a matrix Q, symmetric and positive definite, such that

$$B_1 = Q^2. \tag{3.15}$$

Let $\gamma = \hat{\lambda}^2$, then problem (3.14) is equivalent to eigenvalue problem (3.11). Substituting (3.15) into (3.11), we obtain

$$Q^{-1}B_2Q^{-1}Q\xi = \gamma Q\xi,$$

or

$$Q^{-1}B_2Q^{-1}\eta = \gamma\eta \tag{3.16}$$

with $\eta = Q\xi$. Since $Q^{-1}B_2Q^{-1}$ is symmetric and positive semi-definite, the eigenvalue problem (3.16) has N nonnegative eigenvalues:

$$0 = \gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_N$$

and N real linearly independent eigenvectors $\eta_j, j = 1, 2, \ldots, N$. Thus, problem (3.11) also has these eigenvalues and N real linearly independent eigenvectors $\xi_j = Q^{-1}\eta_j, j = 1, 2, \ldots, N$. Let

$$\hat{\lambda}_j = \sqrt{\gamma_j}, \quad \hat{\mathbf{u}}_{h_\theta}(r) = \sum_{j=1}^{N} \hat{a}_j r^{\hat{\lambda}_j} \xi_j.$$

It is easy to verify that $\hat{\mathbf{u}}_{h_\theta}(r)$ satisfies (3.9), and is bounded as $r \to 0$. Let $r = R$, we obtain (3.13), which is the system satisfied by $\hat{a}_j,\ j = 1, 2, \ldots, N$. □

Lemma 2. *If* $\gamma = 0$ *is an eigenvalue of problem* (3.11), ξ *is the corresponding eigenvector, then*

$$\xi^t\psi(\theta) = \textit{constant}.$$

Proof. Let $v_{h_\theta}(\theta) = \xi^t\psi(\theta)$, since

$$B_1^{-1}B_2\xi = 0 \cdot \xi, \text{ i.e.,} \quad B_2\xi = 0,$$

we get

$$0 = \xi^t B_2 \xi = \int_0^{2\pi} p(\theta)(v'_{h_\theta}(\theta))^2 \, \mathrm{d}\theta,$$

which implies that $v'_{h_\theta}(\theta) = 0, \forall\theta \in [0, 2\pi]$. □

We now introduce a weighted Sobolev space

$$H_*^2(\Omega) = \{v(\mathbf{x}) \colon \|v\|_{2,\Omega}^* < +\infty\}$$

with the weighted norm

$$\|v\|_{2,\Omega}^* = \{\|v\|_{1,\Omega}^2 + \int_\Omega r^2((\partial^2_{x_1x_1}v)^2 + (\partial^2_{x_1x_2}v)^2 + (\partial^2_{x_2x_2}v)^2)\,\mathrm{d}\mathbf{x}\}^{1/2}.$$

Let

$$P_I = \min_{[0,2\pi]}\{p(\theta)\}, \quad P_M = \max_{[0,2\pi]}\{p(\theta)\}.$$

Our main result in this section is the following theorem.

Theorem 2. (i) *Problem* (3.1)–(3.4) *has semi-discrete approximation solution*

$$u_{h_\theta}(r,\theta) = (\hat{\mathbf{u}}_{h_\theta}(r))^t\psi(\theta) \in H^1(\Omega_R), \tag{3.17}$$

where $u_{h_\theta}(r,\theta)$ *is the unique solution of problem* (3.7) *and* (3.8), $\hat{\mathbf{u}}_{h_\theta}(r)$ *is defined in* (3.12).

(ii) *If* $u(\mathbf{x})$, *the exact solution of problem* (3.1)–(3.4), *belongs to* $H_*^2(\Omega_R)$, *then* $u_{h_\theta}(r,\theta)$, *the linear semi-discrete approximation solution with respect to* θ, *satisfies*

$$\|u-u_{h_\theta}\|_{1,\Omega_R} \leqslant ch_\theta\|u\|^*_{2,\Omega_R}, \tag{3.18}$$

where $c=c^*(1+\frac{1}{2}\sqrt{P_M/P_I}\sqrt{4+R^2})\sqrt{4+R^2}$, c^* *is a constant independent of* u,r,R,θ.

The proof of this theorem will be given at the end of this section.

According to this theorem, if we let N_0 denote the number of zero-eigenvalue of problem (3.11), and let

$$\tilde{\lambda}_j=\hat{\lambda}_{N_0+j},\ \tilde{a}_j=\hat{a}_{N_0+j},\ \tilde{f}_j(\theta)=\xi^t_{N_0+j}\psi(\theta),\ j=1,2,\ldots,N-N_0, \tag{3.19}$$

then we obtain the approximation singularity near the singular point of interface problem (2.1)–(2.4):

$$\begin{aligned} u_{h_\theta}(r,\theta) &= \sum_{j=1}^{N_0}\hat{a}_j\xi_j^t\psi(\theta)+\sum_{j=1}^{N-N_0}\tilde{a}_j r^{\tilde{\lambda}_j}\tilde{f}_j(\theta) \\ &= \tilde{c}+\tilde{a}_1 r^{\tilde{\lambda}_1}\tilde{f}_1(\theta)+\cdots+\tilde{a}_{N-N_0}r^{\tilde{\lambda}_{N-N_0}}\tilde{f}_{N-N_0}(\theta), \end{aligned} \tag{3.20}$$

where $\tilde{c}=\sum_{j=1}^{N_0}\hat{a}_j\xi_j^t\psi(\theta)$ is a constant (by Lemma 2). Next, we give two lemmas which will be used in the proof of Theorem 2.

Lemma 3. *For any* $w(\mathbf{x})\in H^1(\Omega_R)$, *under the polar coordinates* (r,θ), *if* $w(R,\theta)=0,\ \forall\theta\in[0,2\pi]$, *then*

$$\|w\|_{1,\Omega_R}\leqslant\sqrt{1+R^2/4}\ |w|_{1,\Omega_R}. \tag{3.21}$$

Proof. Since $w(r,\theta)=-\int_r^R\partial_t w(t,\theta)\,\mathrm{d}t$, using Hölder inequality we get

$$\begin{aligned} \int_0^{2\pi}\int_0^R rw^2(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi}\int_0^R r\left[\int_r^R\partial_t w(t,\theta)\,\mathrm{d}t\right]^2\mathrm{d}r\,\mathrm{d}\theta \\ &\leqslant \int_0^{2\pi}\int_0^R r\left[\int_r^R t^{-1}\,\mathrm{d}t\int_0^R t(\partial_t w(t,\theta))^2\,\mathrm{d}t\right]\mathrm{d}r\,\mathrm{d}\theta \\ &= \frac{R^2}{4}\int_0^{2\pi}\int_0^R r(\partial_r w)^2\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned}$$

Then we have

$$\begin{aligned} \|w\|^2_{1,\Omega_R} &= \int_0^{2\pi}\int_0^R r[w^2+(\partial_r w)^2+r^{-2}(\partial_\theta w)^2]\,\mathrm{d}r\,\mathrm{d}\theta \\ &\leqslant \left(1+\frac{R^2}{4}\right)\int_0^{2\pi}\int_0^R r[(\partial_r w)^2+r^{-2}(\partial_\theta w)^2]\,\mathrm{d}r\,\mathrm{d}\theta=\left(1+\frac{R^2}{4}\right)|w|^2_{1,\Omega_R}, \end{aligned}$$

which gives (3.21). □

For $w \in U$, we denote its linear interpolation with respect to θ by

$$w_\theta^I(r,\theta) = \sum_{i=1}^{N} w(r,\theta_i)\psi_i(\theta) \in U_1^{h_\theta}.$$

We have the following interpolation error estimate.

Lemma 4. *For any $u(\mathbf{x}) \in H_*^2(\Omega_R)$, we have the error estimate*

$$\|u - u_\theta^I\|_{1,\Omega_R} \leqslant c\sqrt{4+R^2}\,h_\theta \|u\|_{2,\Omega_R}^*, \tag{3.22}$$

where c is a constant independent of u, θ, r, R.

Proof. For one-dimension linear interpolation v_θ^I of function $v(\theta)$, let $\rho(\theta) = v(\theta) - v_\theta^I$, we have the well-known estimates:

$$\|\rho\|_{0,[0,2\pi]} \leqslant ch_\theta \|v_\theta\|_{0,[0,2\pi]},$$

$$\|\rho_\theta\|_{0,[0,2\pi]} \leqslant ch_\theta \|v_{\theta\theta}\|_{0,[0,2\pi]},$$

where c is a constant independent of $v(\theta), \theta$. It is easy to check that

$$|\partial_\theta u(r,\theta)| = |-x_2\partial_{x_1}u + x_1\partial_{x_2}u| \leqslant r(|\partial_{x_1}u| + |\partial_{x_2}u|),$$

$$\begin{aligned}|\partial_{\theta\theta}u(r,\theta)| &= |x_2^2\partial_{x_1x_1}u - 2x_1x_2\partial_{x_1x_2}u + x_1^2\partial_{x_2x_2}u - x_1\partial_{x_1}u - x_2\partial_{x_2}u| \\ &\leqslant r^2(|\partial_{x_1x_1}u| + |\partial_{x_1x_2}u| + |\partial_{x_2x_2}u|) + r(|\partial_{x_1}u| + |\partial_{x_2}u|),\end{aligned}$$

$$\begin{aligned}|\partial_{r\theta}u(r,\theta)| = r^{-1}|&-x_1x_2\partial_{x_1x_1}u + (x_1^2 - x_2^2)\partial_{x_1x_2}u + x_1x_2\partial_{x_2x_2}u - x_2\partial_{x_1}u \\ &+x_1\partial_{x_2}u| \leqslant r(|\partial_{x_1x_1}u| + |\partial_{x_1x_2}u| + |\partial_{x_2x_2}u|) + (|\partial_{x_1}u| + |\partial_{x_2}u|).\end{aligned}$$

Hence, inequality (3.22) follows from the following inequality:

$$\begin{aligned}\|u - u_\theta^I\|_{1,\Omega_R}^2 &= \int_0^R r(\|\rho(r,\theta)\|_{0,[0,2\pi]}^2 + \|\partial_r\rho(r,\theta)\|_{0,[0,2\pi]}^2 + r^{-2}\|\partial_\theta\rho(r,\theta)\|_{0,[0,2\pi]}^2)\,\mathrm{d}r \\ &\leqslant ch_\theta^2 \int_0^R\int_0^{2\pi} r[(r^2+4)((\partial_{x_1}u)^2 + (\partial_{x_2}u)^2) \\ &\quad + r^2((\partial_{x_1x_1}u)^2 + (\partial_{x_1x_2}u)^2 + (\partial_{x_2x_2}u)^2)]\,\mathrm{d}\theta\,\mathrm{d}r \\ &\leqslant c(R^2+4)h_\theta^2(\|u\|_{2,\Omega_R}^*)^2. \qquad \square\end{aligned}$$

With these lemmas we can complete the proof of Theorem 2.

Proof of Theorem 2. Since, by Lemma 1, $\hat{\mathbf{u}}_{h_\theta}(r)$ is the solution of problem (3.9) and (3.10), then $u_{h_\theta}(r,\theta) = (\hat{\mathbf{u}}_{h_\theta}(r))^t\psi(\theta)$ is obviously the solution of problem (3.7) and (3.8). Furthermore, expression (3.20) implies that $u_{h_\theta}(r,\theta) \in H^1(\Omega_R)$.

If $u^*_{h_\theta}(r,\theta)\in U_k^{h_\theta}$ is another solution of problem (3.7) and (3.8), the difference $e(r,\theta)=u_{h_\theta}(r,\theta)-u^*_{h_\theta}(r,\theta)$ then satisfies

$$A_1(r\partial_r(r\partial_r e(r,\theta)),v_{h_\theta})-A_2(e(r,\theta),v_{h_\theta})=0,\quad \forall v_{h_\theta}\in V_k^{h_\theta},$$

$$e(r,\theta)|_{r=R}=0,\quad e(r,\theta)\ \text{is bounded as}\ r\to 0.$$

Multiply the above equation by $-r^{-1}$, let $v_{h_\theta}=e(r,\theta)$, integrate from 0 to R, and notice the fact that

$$r\partial_r w(r,\theta)\to 0,\ \text{as}\ r\to 0,\quad \forall w(r,\theta)\in\ H^1(\Omega_R), \tag{3.23}$$

we obtain

$$\begin{aligned}0&=-\int_0^{2\pi}\int_0^R p(\theta)\partial_r(r\partial_r e)e\,\mathrm{d}r\,\mathrm{d}\theta+\int_0^{2\pi}\int_0^R r^{-1}p(\theta)(\partial_\theta e)^2\,\mathrm{d}r\,\mathrm{d}\theta\\&=\int_0^{2\pi}\int_0^R r\,p(\theta)(\partial_r e)^2\,\mathrm{d}r\,\mathrm{d}\theta+\int_0^{2\pi}\int_0^R r^{-1}p(\theta)(\partial_\theta e)^2\,\mathrm{d}r\,\mathrm{d}\theta,\end{aligned}$$

this equality and the boundary condition ensure that $e(r,\theta)=0$ in Ω_R, namely, the uniqueness of solution of problem (3.7) and (3.8).

To show the validity of (3.18), let

$$e(r,\theta)=u(r,\theta)-u_{h_\theta}(r,\theta)=(u-u_\theta^I)+(u_\theta^I-u_{h_\theta})=\rho(r,\theta)+\sigma(r,\theta).$$

From (3.5)–(3.8), $e(r,\theta)$ satisfies

$$A_1(r\partial_r(r\partial_r e(r,\theta)),v_{h_\theta})-A_2(e(r,\theta),v_{h_\theta})=0,\quad \forall v_{h_\theta}\in V_k^{h_\theta}.$$

Multiply the above equation by $-r^{-1}$, let $v_{h_\theta}=\sigma(r,\theta)$, and integrate from 0 to R, we get

$$\begin{aligned}&-\int_0^{2\pi}\int_0^R p(\theta)\partial_r(r\partial_r\sigma)\sigma\,\mathrm{d}r\,\mathrm{d}\theta+\int_0^{2\pi}\int_0^R r^{-1}p(\theta)(\partial_\theta\sigma)^2\,\mathrm{d}r\,\mathrm{d}\theta\\&=\int_0^{2\pi}\int_0^R p(\theta)\partial_r(r\partial_r\rho)\sigma\,\mathrm{d}r\,\mathrm{d}\theta-\int_0^{2\pi}\int_0^R r^{-1}p(\theta)\partial_\theta\rho\partial_\theta\sigma\,\mathrm{d}r\,\mathrm{d}\theta.\end{aligned} \tag{3.24}$$

Integrating by parts once, using (3.23) and noticing that $\sigma(R,\theta)=0$, the left-hand side (LHS) of the above equality becomes

$$\mathrm{LHS}=\int_0^{2\pi}\int_0^R r\,p((\partial_r\sigma)^2+r^{-2}(\partial_\theta\sigma)^2)\,\mathrm{d}r\,\mathrm{d}\theta.$$

Similarly, the right-hand side (RHS) of (3.24) becomes

$$\begin{aligned}\mathrm{RHS}&=-\int_0^{2\pi}\int_0^R r\,p\partial_r\rho\partial_r\sigma\,\mathrm{d}r\,\mathrm{d}\theta-\int_0^{2\pi}\int_0^R r^{-1}p\partial_\theta\rho\partial_\theta\sigma\,\mathrm{d}r\,\mathrm{d}\theta\\&\leqslant\frac12\int_0^{2\pi}\int_0^R r\,p((\partial_r\rho)^2+r^{-2}(\partial_\theta\rho)^2)\,\mathrm{d}r\,\mathrm{d}\theta+\frac12\,\mathrm{LHS}.\end{aligned}$$

Therefore, we have

$$P_I|\sigma|^2_{1,\Omega_R} \leqslant \int_0^{2\pi}\int_0^R rp((\partial_r\sigma)^2 + r^{-2}(\partial_\theta\sigma)^2)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$\leqslant \int_0^{2\pi}\int_0^R rp((\partial_r\rho)^2 + r^{-2}(\partial_\theta\rho)^2)\,\mathrm{d}r\,\mathrm{d}\theta \leqslant P_M\|\rho\|^2_{1,\Omega_R}.$$

Using Lemma 3, we have

$$\|e\|_{1,\Omega_R} \leqslant \|\sigma\|_{1,\Omega_R} + \|\rho\|_{1,\Omega_R} \leqslant (1 + \tfrac{1}{2}\sqrt{P_M/P_I}\sqrt{4+R^2})\|\rho\|_{1,\Omega_R}.$$

Then (3.18) follows from the above inequality and Lemma 4. □

Remark 1. According to Theorem 2, the approximation of the singularity expressed by (3.20) near the singular point can be obtained only by solving a simple eigenvalue problem (3.11), which is one dimension less than the original problem, the computational work is trivial compared to the work for the finite element method for the whole problem.

Remark 2. If $f(\mathbf{x})$ is not identically zero in a neighborhood of the origin, then corresponding to (3.9), we have the following nonhomogeneous problem,

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}(B_1\hat{\mathbf{u}}_{h_\theta}(r))\right) - B_2\hat{\mathbf{u}}_{h_\theta}(r) = \mathbf{F}(r) = -r^2\int_0^{2\pi} f(r,\theta)\psi(\theta)\,\mathrm{d}\theta, \tag{3.25}$$

$$\hat{\mathbf{u}}_{h_\theta}(r)|_{r=R} = \hat{\mathbf{u}}^0_{h_\theta},\ \ \hat{\mathbf{u}}_{h_\theta}(r) \text{ is bounded as } r\to 0. \tag{3.26}$$

Using (3.12) we can obtain theoretically a special solution of (3.25) by the method of variational parameters. But for general $f(r,\theta)$, the singular property of this solution cannot be obtained easily. However, (3.25) can be solved for some special cases. For example, if $r^2 f(r,\theta)$ can be expanded as the following form:

$$r^2 f(r,\theta) = \sum_{j=1}^{J} r^{\alpha_j} g_j(\theta) + r^\beta f_1(r,\theta), \quad (r,\theta)\in\Omega_R, \tag{3.27}$$

where β and each α_j are constants, $\beta \geqslant 3$, $0<\alpha_1<\alpha_2<\cdots<\alpha_J<3$, each α_j is not the eigenvalue of $B_1^{-1}B_2$, and $f_1(r,\theta)$ is continuous in $[0,R]$ as a function of r, then we have

$$\mathbf{F}(r) = \sum_{j=1}^{J}\mathbf{d}_j r^{\alpha_j} + r^\beta\mathbf{F}_1(r), \tag{3.28}$$

where $\mathbf{d}_j$, $j=1,\ldots,J$, are constant vectors, and $\mathbf{F}_1(r)\in C[0,R]$. We consider the system

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}(B_1\mathbf{w}(r))\right) - B_2\mathbf{w}(r) = r^\beta\mathbf{F}_1(r), \quad 0<r\leqslant R. \tag{3.29}$$

Let $\mathbf{w}(r) = r^{\beta-1}\mathbf{w}_1(r)$, then $\mathbf{w}_1(r)$ satisfies

$$r^2\mathbf{w}_1''(r) + (2\beta-1)r\mathbf{w}_1'(r) + [(\beta-1)^2 I - B_1^{-1}B_2]\mathbf{w}_1(r) = rB_1^{-1}\mathbf{F}_1(r). \tag{3.30}$$

Let

$$\mathbf{v}_1(r)=\mathbf{w}_1(r),\quad \mathbf{v}_2(r)=r\mathbf{w}_1'(r)-\mathbf{w}_1(r),\quad \mathbf{v}(r)=\begin{bmatrix}\mathbf{v}_1(r)\\ \mathbf{v}_2(r)\end{bmatrix}.$$

Then (3.30) can be written in the following form:

$$\mathbf{v}'(r)-\frac{B}{r}\mathbf{v}(r)=\mathbf{g}(r),\quad 0<r\leqslant R, \tag{3.31}$$

where

$$B=\begin{bmatrix} I & I\\ -\beta^2 I+B_1^{-1}B_2 & -(2\beta-1)I\end{bmatrix},\quad \mathbf{g}(r)=\begin{bmatrix}0\\ B_1^{-1}\mathbf{F}_1(r)\end{bmatrix}\in C[0,R].$$

System (3.31) has a special solution $\tilde{\mathbf{v}}(r)\in C[0,R]$ [7]. Then (3.29) has a special solution $\tilde{\mathbf{w}}(r)=r^{\beta-1}\tilde{\mathbf{v}}_1(r)$ and it is easy to check that $\tilde{\mathbf{w}}(r)\in C^2[0,R]$.

On the other hand, by the method of undetermined coefficients we can easily obtain the special solution $\mathbf{w}(r)=(\alpha_j^2 B_1-B_2)^{-1}\mathbf{d}_j r^{\alpha_j}$ corresponding to the right-hand side $\mathbf{d}_j r^{\alpha_j}$, which gives the singularity information needed for the singular finite element space. □

4. The singular finite element approximation

We consider the singular finite element approximation of problem (2.1)–(2.4). Assume that J_h is a quasi-uniform triangulation of Ω and the resulted meshes resolve the interfaces. Let

$$\Omega_h=\bigcup_{K\in J_h}K,\quad \Gamma_h=\partial\Omega_h,$$

where K is a triangle and

$$h_K/\rho_K\leqslant\sigma,\quad \forall K\in J_h,$$

where $\sigma>0$ is a constant, h_K=diameter of K, ρ_K=diameter of the inscribed circle of K, and $h=\max\{h_K, K\in J_h\}$. Let $\{\mathbf{x}_s,\ s=1,2,\dots,n\}$ be the node set, N_b be the set of index of the boundary nodes. For simplicity, we assume that $\Omega_h=\bar{\Omega}$ and then $\Gamma_h=\Gamma$.

According to Theorem 2, problem (2.1)–(2.4) has the approximate singularity expression (3.20), therefore, we can construct the singular finite element space

$$S_k^h=\mathrm{Span}\{\varphi_i(\mathbf{x}),\eta(r,\theta)r^{\tilde{\lambda}_j}\tilde{f}_j(\theta), i=1,2,\dots,n,\ j=1,2,\dots,m\}, \tag{4.1}$$

where the standard finite element basis functions $\{\varphi_i(\mathbf{x})\}_1^n$ are piecewise polynomials of degree k, $\eta(r,\theta)$ is the proper smooth cut-off function which equals one at the origin and vanishes on the boundary Γ_h. Let

$$U_k^h=\{u_h(\mathbf{x}): u_h(\mathbf{x})\in S_k^h, u_h(\mathbf{x}_s)=g(\mathbf{x}_s),\ \forall s\in N_b\},$$

$$V_k^h=\{v_h(\mathbf{x}): v_h(\mathbf{x})\in S_k^h, v_h(\mathbf{x}_s)=0,\ \forall s\in N_b\}\subset H_0^1(\Omega).$$

Then the finite element approximation of problem (2.5) is

$$\text{Find } u_h(\mathbf{x}) \in U_k^h, \text{ such that}$$
$$a(u_h, v_h) = (f, v_h), \ \forall v_h(\mathbf{x}) \in V_k^h. \tag{4.2}$$

Obviously, this variation problem has a unique solution $u_h(\mathbf{x}) \in U_k^h$.

Next, we consider the accuracy of $u_h(\mathbf{x}) \in U_1^h$. We denote the semi-discrete step length introduced in Section 3 by $h_\theta = \max_{1 \leqslant i \leqslant N} |\theta_{i+1} - \theta_i|$, and introduce the Sobolev spaces

$$H_{**}^2(\Omega) = \{v(\mathbf{x}) : \|v\|_{2,\Omega}^{**} < +\infty\}, \ H_\sharp^2(\Omega) = \{v(\mathbf{x}) : \|v\|_{2,\Omega}^\sharp < +\infty\},$$

where

$$\|v\|_{2,\Omega}^{**} = \left\{ \|v\|_{1,\Omega}^2 + \sum_{i=1}^N |v|_{2,\Omega_i^{**}}^2 \right\}^{1/2}, \quad \Omega_i^{**} = \Omega \cap \{(r,\theta) : \theta_i \leqslant \theta \leqslant \theta_{i+1}, r \geqslant 0\},$$

$$\|v\|_{2,\Omega}^\sharp = \left\{ \|v\|_{1,\Omega}^2 + \sum_{j=1}^M |v|_{2,\Omega_j}^2 \right\}^{1/2}, \quad \Omega_j = \Omega \cap \{(r,\theta) : \phi_j \leqslant \theta \leqslant \phi_{j+1}, r \geqslant 0\}.$$

According to Theorem 2 and the approximate singularity expression (3.20),

$$u(r,\theta) \approx u_{h_\theta}(r,\theta) = \sum_{j=1}^m \tilde{a}_j r^{\tilde{\lambda}_j} \tilde{f}_j(\theta) + \sum_{j=m+1}^{N-N_0} \tilde{a}_j r^{\tilde{\lambda}_j} \tilde{f}_j(\theta) + \tilde{c} \equiv \tilde{w}_0 + \tilde{w}, \ (r,\theta) \in \Omega_R,$$

where $0 < \tilde{\lambda}_1 \leqslant \cdots \leqslant \tilde{\lambda}_m \leqslant 1, \ \tilde{\lambda}_{N-N_0} \geqslant \cdots \geqslant \tilde{\lambda}_{m+1} > 1$, hence

$$\tilde{w}_0 \equiv \sum_{j=1}^m \tilde{a}_j r^{\tilde{\lambda}_j} \tilde{f}_j(\theta) \in H^1(\Omega), \quad \tilde{w} \in H_{**}^2(\Omega), \text{ but } \tilde{w}_0 \bar{\in} H_{**}^2(\Omega).$$

Writing the exact solution of problem (2.1)–(2.4) in the following form:

$$u(r,\theta) \equiv w_0 + w, \quad w_0 \in H^1(\Omega), \quad w \in H_\sharp^2(\Omega), \text{ but } w_0 \bar{\in} H_\sharp^2(\Omega),$$

and for any smooth cut-off function $\eta(r,\theta)$ used in Eq. (4.1) with $k = 1$, we denote

$$\Omega_R^* = \begin{cases} \text{empty set} & \text{if } \operatorname{Supp}\{\eta\} \equiv \overline{\{\mathbf{x} : \eta(\mathbf{x}) \neq 0\}} \subset \Omega_R, \\ \operatorname{Supp}\{\eta\} \setminus \Omega_R & \text{otherwise.} \end{cases}$$

Then we have

Theorem 3. *If the singular finite element space S_k^h is selected such that $k = 1$ and $\eta(r,\theta)$ satisfies*

$$\eta(0,\theta) = 1, \quad \eta(r,\theta) = 0 \text{ on } \Gamma_h, \quad [1 - \eta(r,\theta)]u \in H_\sharp^2(\Omega), \tag{4.3}$$

then we have the error estimate

$$\|u - u_h\|_{1,\Omega} \leqslant c_\eta h_\theta \|u\|_{2,\Omega_R}^* + ch\{\|(1-\eta)u\|_{2,\Omega}^\sharp + \|\eta u\|_{2,\Omega_R^*}^\sharp\}$$
$$+ ch\{\|\eta \tilde{w}\|_{2,\Omega_R}^{**} + \|\eta \tilde{w}_0\|_{2,\Omega_R^*}^{**}\}, \tag{4.4}$$

where $c_\eta = c \max_{\Omega_R}\{|\nabla \eta|, |\eta|\}$, c is a constant independent of h_θ and h.

Proof. We define two restriction functions $\zeta(\mathbf{x})$ and $\zeta^*(\mathbf{x})$ as

$$\zeta(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega_R, \\ 0 & \text{otherwise,} \end{cases} \qquad \zeta^*(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega_R^*, \\ 0 & \text{otherwise,} \end{cases}$$

and take the interpolation of $u(\mathbf{x})$ as

$$u_I = \eta(r,\theta) \sum_{j=1}^{m} \tilde{a}_j r^{\tilde{\lambda}_j} \tilde{f}_j(\theta) + v_I^*, \qquad v_I^* = \sum_{s=1}^{n} v(\mathbf{x_s}) \varphi_s(\mathbf{x}),$$

where $v = \zeta^* \eta u + (1-\eta)u + \zeta\eta\tilde{w} - \zeta^*\eta\tilde{w}_0$.

It is easy to check that $u_I \in U_1^h$, and

$$u - u_I = (\zeta + \zeta^*)\eta u + (1-\eta)u - u_I = \zeta\eta(u - u_{h_\theta}) + \{\zeta^*\eta u - [\zeta^*\eta u]_I^*\}$$
$$+\{(1-\eta)u - [(1-\eta)u]_I^*\} + \{\zeta\eta\tilde{w} - [\zeta\eta\tilde{w}]_I^*\} - \{\zeta^*\eta\tilde{w}_0 - [\zeta^*\eta\tilde{w}_0]_I^*\}.$$

Notice that v_I^* is just the standard piecewise linear interpolation of $v(\mathbf{x})$, we thus have error estimate

$$\|u - u_I\|_{1,\Omega} \leqslant \|\eta(u - u_{h_\theta})\|_{1,\Omega_R} + ch\{\|\eta u\|^{\sharp}_{2,\Omega_R^*} + \|(1-\eta)u\|^{\sharp}_{2,\Omega}\}$$
$$+ch\{\|\eta\tilde{w}\|^{**}_{2,\Omega_R} + \|\eta\tilde{w}_0\|^{**}_{2,\Omega_R^*}\}. \tag{4.5}$$

Let

$$e(\mathbf{x}) = u(\mathbf{x}) - u_h(\mathbf{x}) = (u - u_I) + (u_I - u_h) \equiv \rho + \sigma,$$

from (2.5) and (4.2), $e(\mathbf{x})$ satisfies

$$0 = a(e, v_h) = a(\rho + \sigma, v_h), \ \forall v_h(\mathbf{x}) \in V_1^h. \tag{4.6}$$

Take $v_h = \sigma = u_I - u_h \ \in V_1^h$, then

$$a(\sigma,\sigma) = -a(\rho,\sigma) \leqslant \tfrac{1}{2}a(\rho,\rho) + \tfrac{1}{2}a(\sigma,\sigma).$$

Since $\sigma \in H_0^1(\Omega)$, using Poincarè Friedrichs inequality and the above inequality we obtain

$$\|\sigma\|_{1,\Omega} \leqslant c|\sigma|_{1,\Omega} \leqslant c\|\rho\|_{1,\Omega}.$$

Therefore, error estimate (4.4) follows from (4.5), (3.18) and the following inequality:

$$\|e\|_{1,\Omega} \leqslant \|\rho\|_{1,\Omega} + \|\sigma\|_{1,\Omega} \leqslant c\|\rho\|_{1,\Omega}. \quad \square$$

We can easily choose $\eta(r,\theta)$ satisfying (4.3). For example, for any $0 < \alpha < \beta \leqslant$ radius of the inscribed circle of Ω, the following function is in $C^2(\Omega)$ and satisfies (4.3):

$$\eta(r,\theta) = \begin{cases} 1 & \text{if } 0 \leqslant r < \alpha, \\ -\dfrac{(r-\alpha)^3}{(\beta-\alpha)^5}[6r^2 + 3(\alpha - 5\beta)r + \alpha^2 - 5\alpha\beta + 10\beta^2] + 1 & \text{if } \alpha \leqslant r < \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3. A common disadvantage of using global basis functions $\eta(r,\theta)r^{\tilde{\lambda}_j}\tilde{f}_j(\theta), j=1,2,\ldots,m$, is that the bandwidth of the stiffness matrix is much larger than the conventional one. Of course, the smaller support of $\eta(r,\theta)$ will yield a relatively smaller bandwidth. However, if the support is too small, the finite element solution will be polluted by the effect of the singularities of exact solution. Therefore, in practice, we should make the balance between the accuracy of finite element solution and the computation cost. Also, the unknowns should be arranged such that the stiffness matrix is close to the standard type.

Obviously, Theorem 3 implies that

$$\lim_{h\to 0}\|u-u_h\|_{1,\Omega} \leqslant c_\eta h_\theta,$$

namely, u_h is convergent. Comparing $u_{h_\theta}(r,\theta)$ with $u(r,\theta)$ in Ω_R, we can expect that $\tilde{w}_0$ and $\tilde{w}$ are the corresponding approximation of w_0 and w, respectively. Thus, if we assume that

$$\|w-\tilde{w}\|_{1,\Omega_R} \leqslant ch_\theta, \tag{4.7}$$

where c is a constant independent of h_θ, then we have the following theorem.

Theorem 4. *If assumption (4.7) holds and the smooth cut-off function $\eta(r,\theta)$ is chosen such that*

$$\eta(0,\theta)=1, \quad \operatorname{supp}\{\eta\} \subseteq \Omega_R, \quad [1-\eta(r,\theta)]u \in H^2_\sharp(\Omega), \tag{4.8}$$

then the singular finite element solution $u_h(\mathbf{x}) \in U_1^h$ satisfies:

$$\|u-u_h\|_{1,\Omega} \leqslant c(h_\theta + h), \tag{4.9}$$

where c is a constant independent of h_θ and h.

Proof. Similar to the proof of Theorem 3, here we only make the following changes:

$$v=(1-\eta)u+\eta w,$$

hence

$$\begin{aligned} u-u_I &= \eta(u-u_{h_\theta}) + \{(1-\eta)u - [(1-\eta)u]_I^*\} \\ &\quad + \{\eta\tilde{w} - \eta w\} + \{\eta w - [\eta w]_I^*\}, \end{aligned}$$

and

$$\begin{aligned} \|u-u_I\|_{1,\Omega} &\leqslant \|\eta(u-u_{h_\theta})\|_{1,\Omega_R} + \|\eta(\tilde{w}-w)\|_{1,\Omega_R} \\ &\quad + ch\{\|(1-\eta)u\|^\sharp_{2,\Omega} + \|\eta w\|^\sharp_{2,\Omega_R}\} \leqslant c(h_\theta+h). \qquad \square \end{aligned}$$

5. Numerical examples

Let $\Omega=\{\mathbf{x}=(x_1,x_2): -1<x_1<1, -1<x_2<1\}$ with boundary Γ, consider the following interface problem:

$$-\nabla(p\nabla u)=0 \quad \text{in } \Omega,$$

$$u=g \quad \text{on } \Gamma,$$

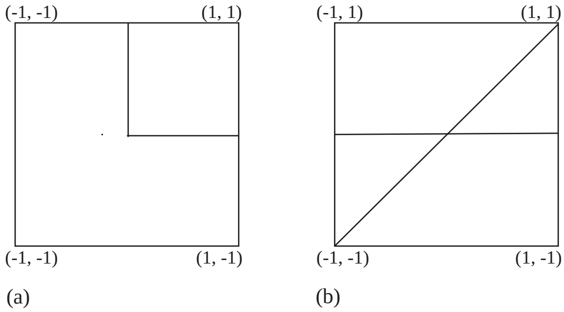

Fig. 2. (a) Domain with two interfaces; (b) Domain with four interfaces.

$$u(r,\phi_k - 0) = u(r,\phi_k + 0), \quad 1 \leqslant k \leqslant M,$$

$$p_{k-1}\frac{\partial u}{\partial \mathbf{n}}(r,\phi_k - 0) = -p_k\frac{\partial u}{\partial \mathbf{n}}(r,\phi_k + 0), \quad 1 \leqslant k \leqslant M.$$

We use the piecewise linear polynomial with respect to θ to compute the approximation of the singularity discussed in Section 3, and take the piecewise linear polynomial with respect to (x_1,x_2) as the standard finite element basis functions used in Eq. (4.1) with $k=1$.

In order to reduce the pollution of the singularity, here we choose the smooth cut-off function of large support as the following:

$$\eta(\mathbf{x}) = (1-x_1^2)(1-x_2^2), \ \mathbf{x}\in\bar{\Omega}, \ \text{i.e.,} \ \eta(r,\theta) = 1 - r^2 + \frac{1}{4}r^4\sin^2(2\theta), \ \text{in} \ \bar{\Omega}.$$

We can easily check that $\eta(r,\theta)$ satisfies (4.3). As comparison, we have also computed the approximate solution u_h^* of problem (2.1)–(2.4) by the standard linear finite element method.

Example 1. Let $\phi_1 = 0, \ \phi_2 = \pi/2$,

$$p = \begin{cases} \sqrt{2}+2 & 0 \leqslant \theta \leqslant \dfrac{\pi}{2}, \\ \sqrt{2}-2 & \dfrac{\pi}{2} \leqslant \theta \leqslant 2\pi, \end{cases}$$

$$g(r,\theta) = \begin{cases} r^a(\cos(a\theta) + c_1\sin(a\theta)), & 0 \leqslant \theta \leqslant \dfrac{\pi}{2}, \\ r^a(c_2\cos(a\theta) + c_3\sin(a\theta)), & \dfrac{\pi}{2} \leqslant \theta \leqslant 2\pi, \end{cases}$$

where $a=0.5, \ c_1=\sqrt{2}-1, \ c_2=-1, \ c_3=\sqrt{2}+1$. The exact solution of this problem is $u(r,\theta)=g(r,\theta)$. This problem has two interfaces at $\theta=0$ and $\theta=\pi/2$ (Fig. 2(a)). Three meshes (mesh A_θ, mesh B_θ, mesh C_θ) of interval $[0,2\pi]$ are used to compute the approximate singularity, they are uniform grids with steps $h_\theta = \pi/80, \pi/160, \pi/320$. The first ten eigenvalues of problem (3.11) are listed in Table 1. We can see that the eigenvalues converge as $h_\theta \to 0$.

Table 1
Discrete eigenvalues

Mesh	A_θ	B_θ	C_θ
γ_1	0.1061262D − 11	0.2434997D − 11	0.1909903D − 10
γ_2	0.2500000D + 00	0.2500000D + 00	0.2500000D + 00
γ_3	0.2250000D + 01	0.2250000D + 01	0.2250000D + 01
γ_4	0.3999998D + 01	0.4000000D + 01	0.4000000D + 01
γ_5	0.3999998D + 01	0.4000000D + 01	0.4000000D + 01
γ_6	0.6249994D + 01	0.6250000D + 01	0.6250000D + 01
γ_7	0.1224995D + 02	0.1225000D + 02	0.1225000D + 02
γ_8	0.1599989D + 02	0.1599999D + 02	0.1600000D + 02
γ_9	0.1599989D + 02	0.1599999D + 02	0.1600000D + 02
γ_{10}	0.2024978D + 02	0.2024999D + 02	0.2025000D + 02

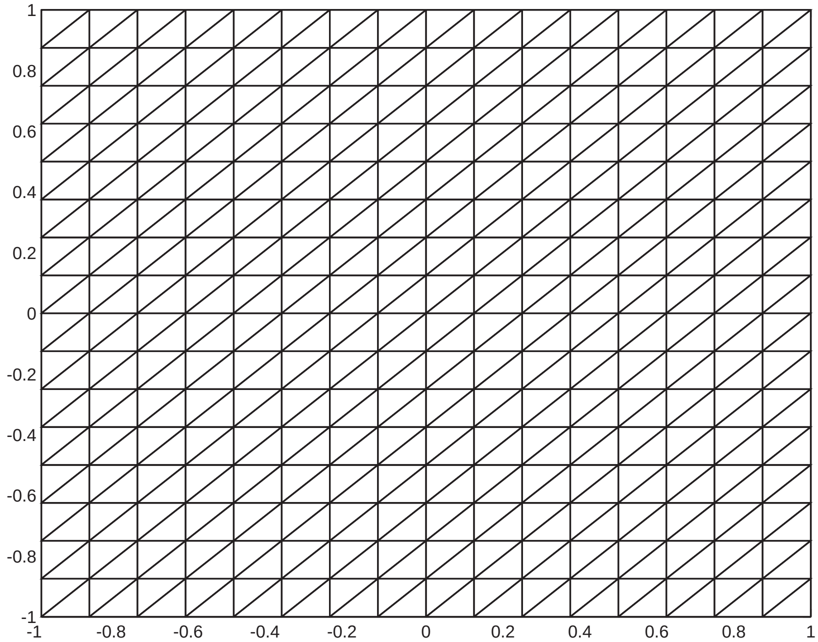

Fig. 3. A conventional mesh: mesh A.

According to (2.9) and Theorem 2, we take $m=1$ and need only $r^{\tilde{\lambda}_1}\tilde{f}_1(\theta)$, i.e., $r^{0.5}\tilde{f}_1(\theta)$, together with the standard finite element basis functions to construct the singular finite element space:

$$S_1^h = \mathrm{Span}\{\varphi_i(\mathbf{x}), \eta(r,\theta)r^{\tilde{\lambda}_1}\tilde{f}_1(\theta),\ i=1,2,\ldots,n\}, \tag{5.1}$$

where $\{\varphi_i(\mathbf{x})\}_1^n$ are the standard linear finite element basis functions.

For three meshes (mesh A, mesh B, mesh C, mesh A shown in Fig. 3) of $\bar{\Omega}$, which are uniform grids with steps $h = h_A, h_A/2, h_A/4$, the errors of the approximate solutions are shown in Table 2.

Table 2
The errors of approximation solutions

Mesh	Mesh A	Mesh B	Mesh C
$\|u-u_h\|_{L^2(\Omega)}$	0.246D − 01	0.767D − 02	0.219D − 02
$\|u-u_h^*\|_{L^2(\Omega)}$	0.634D − 01	0.317D − 01	0.158D − 01
$\|u-u_h\|_{H^1(\Omega)}$	0.398D − 00	0.221D − 00	0.116D − 00
$\|u-u_h^*\|_{H^1(\Omega)}$	0.634D − 00	0.443D − 00	0.311D − 00

Table 3
The convergence rate

	$u-u_h$	$u-u_h^*$
In $\|\cdot\|_{L^2(\Omega)}$ norm:	$O(h^{1.8})$	$O(h^{1.0})$
In $\|\cdot\|_{H^1(\Omega)}$ norm:	$O(h^{0.9})$	$O(h^{0.5})$

Table 3 shows the order of convergence for the approximate solutions. We can see that the singular finite element solution u_h has much higher accuracy than the standard linear finite element solution u_h^*.

Example 2 (Wu and Han [20]). Let $\phi_1=0,\ \phi_2=\pi/4,\ \phi_3=\pi,\ \phi_4=5\pi/4$,

$$p=\begin{cases} -\tan((\omega-c)a), & 0\leqslant\theta\leqslant\dfrac{\pi}{4},\\[2mm] \tan(ba), & \dfrac{\pi}{4}\leqslant\theta\leqslant\pi,\\[2mm] -\tan(ca), & \pi\leqslant\theta\leqslant\dfrac{5\pi}{4},\\[2mm] -\tan((\phi-b)a), & \dfrac{5\pi}{4}\leqslant\theta\leqslant 2\pi,\end{cases}$$

and

$$g(r,\theta)=\begin{cases} r^a\cos((\omega-c)a)\cos((\theta-\phi+b)a), & 0\leqslant\theta\leqslant\dfrac{\pi}{4},\\[2mm] r^a\cos(ba)\cos((\theta-\pi+c)a), & \dfrac{\pi}{4}\leqslant\theta\leqslant\pi,\\[2mm] r^a\cos(ca)\cos((\theta-\pi-b)a), & \pi\leqslant\theta\leqslant\dfrac{5\pi}{4},\\[2mm] r^a\cos((\phi-b)a)\cos((\theta-\phi-\pi-c)a), & \dfrac{5\pi}{4}\leqslant\theta\leqslant 2\pi,\end{cases}$$

where $a=0.01,\ b=0.1,\ c=-199\pi/4,\ \phi=\pi/4$ and $\omega=3\pi/4$. This problem has four interfaces at $\theta=0,\ \theta=\pi/4,\ \theta=\pi$ and $\theta=5\pi/4$ (Fig. 2(b)), the exact solution of this problem is $u(r,\theta)=g(r,\theta)$. Similar to Example 1, three meshes (mesh A_θ, mesh B_θ, mesh C_θ) of interval $[0,2\pi]$ are used to

Table 4
Discrete eigenvalues

Mesh	A_θ	B_θ	C_θ
γ_1	0.0	0.0	0.0
γ_2	0.1000000D − 03	0.1000000D − 03	0.1000000D − 03
γ_3	0.1777746D + 01	0.1777746D + 01	0.1777746D + 01
γ_4	0.1777930D + 01	0.1777931D + 01	0.1777931D + 01
γ_5	0.7110796D + 01	0.7110805D + 01	0.7110805D + 01
γ_6	0.7111165D + 01	0.7111174D + 01	0.7111175D + 01
γ_7	0.1591999D + 02	0.1592009D + 02	0.1592010D + 02
γ_8	0.1599989D + 02	0.1599999D + 02	0.1600000D + 02
γ_9	0.1599989D + 02	0.1599999D + 02	0.1600000D + 02
γ_{10}	0.1607999D + 02	0.1608009D + 02	0.1608010D + 02

Table 5
The errors of approximation solutions

Mesh	Mesh A	Mesh B	Mesh C
$\|u - u_h\|_{L^2(\Omega)}$	0.767D − 04	0.199D − 04	0.500D − 05
$\|u - u_h^*\|_{L^2(\Omega)}$	0.321D − 02	0.268D − 02	0.230D − 02
$\|u - u_h\|_{H^1(\Omega)}$	0.253D − 02	0.125D − 02	0.617D − 03
$\|u - u_h^*\|_{H^1(\Omega)}$	0.422D − 01	0.429D − 01	0.420D − 01

Table 6
The convergence rate

	$u - u_h$	$u - u_h^*$
In $\|\cdot\|_{L^2(\Omega)}$ norm:	$O(h^{1.98})$	$O(h^{0.26})$
In $\|\cdot\|_{H^1(\Omega)}$ norm:	$O(h^{1.0})$	$O(h^{0.05})$

compute the approximate singularity and three meshes (mesh A, mesh B, mesh C, mesh A shown in Fig. 3) of $\bar{\Omega}$ are used to compute the approximate solutions. The first 10 eigenvalues of problem (3.11) are listed in Table 4.

Here we also need only $r^{\tilde{\lambda}_1}\tilde{f}_1(\theta)$, i.e., $r^{0.01}\tilde{f}_1(\theta)$, together with the standard linear finite element basis functions to construct the singular finite element space S_1^h. The errors and convergence rate of the approximate solutions are shown in Tables 5 and 6, respectively. We can see that the singular finite element solution u_h has almost optimal accuracy, but the accuracy of standard linear finite element solution u_h^* is nasty.

6. Conclusions

The singular finite element method is applied to the interface problem. The singularity of the solution is obtained approximately by solving an eigenvalue problem. Under proper assumptions the optimal error estimate is obtained between the finite element solution and the exact solution. Numerical examples show that the method works well, gives very good numerical results, and verifies the convergence results.

Acknowledgements

The authors wish to thank the referees for many valuable suggestions.

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